International Journals of Science and Innovation Technology
Vol. 1. Iss. 1 2014 PP. 001-012
http://www.seitejournals.com/

Thesis

Some New Hermite-Hadamard Integral Inequalities for Convex functions

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Received July 5, 2014, Accepted July 30, 2014, Published August 5th 2014

Abstract

In this paper, we obtain some new integral inequalities of Hermite-Hadamard type involving two newly introduced convex functions, using a simple analytical technique. A few number of results for special means of real numbers are also deduced for applications.

Key words: Hermite-Hadamard's inequality, Convex function, Godunova-Levin function, MT-convex function, m-Godunova-Levin function, m-MT-convex function.

INTRODUCTION

Let $f : I \subseteq R \to R$ be a convex function defined on the interval $I = [c, d]$ of the real numbers and let $a, b \in [c, d]$ where $a < b$. Then, the following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \qquad (1)$$

is known in the literature as the Hermite-Hadamard integral inequality (Hadamard 1893). The double inequality (1), which can be said to be the first fundamental result for convex functions with a natural geometric interpretation and many applications, has attracted and continues to attract much interest

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in the field of Mathematical Sciences and Engineering (Godunova and Levin 1985, Dragomir et al. 1995, Bakula et al. 2006, Tunç 2010 and Omotoyinbo and Mogbademu 2013).

Both inequalities hold in the reversed direction if *f* is concave. Since its discovery in 1881, a number of research papers and texts have been written on it, providing new proofs, noteworthy extensions, generalizations and numerous applications (Godunova and Levin 1985, Dragomir et al. 1995, Bakula et al. 2006, Tunç 2010).

Convexity is a natural and powerful property of functions that play a significant role in many areas of mathematics, both pure and applied. It ties together notions from topology, algebra, geometry and analysis and is an important tool in optimization, control theory, mathematical programming and game theory.

Some definitions relating to convex functions are given below.

Definition 1.1 (Toader 1988). A function $f{:}I \longrightarrow R$ defined on a convex set X is called convex if for every $x, y \in I$ and $t \in [0,1]$, we have

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y) \quad (2)$$

Definition 1.2 (Bakula et al. 2006). A function $f{:}\ [0,b] \longrightarrow R$ is said to be *m-convex, if* for every $x, y \in [0,b]$, $b > 0$ and $t \in [0,1]$, we have

$$f(tx+m(1-t)y) \leq tf(x)+m(1-t)f(y), \text{ where } m \in [0,1]. \quad (3)$$

Remark 1.3. For $m = 1$ in(3), we recapture the concept of convex functions defined on $[0,b]$ and, for$m = 0$, the concept of star-shaped functions defined on $[0, b]$ is obtained.

Definition 1.4 (Godunova-Levin 1985). A function $f{:}I \longrightarrow R$ is Godunova-Levin function or said to belong to the class $Q(I)$ if f is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the inequality;

$$f(tx+(1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t} \quad . \quad (4)$$

Definition 1.5 (Tunc and Yildirim 2012). A function $f{:}\ I \subseteq R \longrightarrow R$ is said to belong to the class $MT(I)$ if it is nonnegative and $\forall x, y \in I$ and $t \in (0,1)$, satisfies the inequality

$$f(tx+(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y) \quad . \quad (5)$$

Clearly, $Q(I) \supset MT(I)$.

Definition 1.6 (Varošanec 2007). If $f, g{:}\ X \longrightarrow R$ are two functions, for $x, y \in X$. Then, the following inequality

$$(f(x)+f(y))\,(g(x)+g(y)) \geq 0, \quad (6)$$

is said to be similarly ordered (briefly s. o.) for the two functions.

Many researchers have, in the last few decades, devoted their efforts to apply the Hermite-

Hadamard inequality (1.1) for different classes of convex functions.

Dragomir et al. (1995) obtained the following two new inequalities of Hadamard type for classes of Godunova-Levin functions and *P*-functions.

Theorem 1.7 (Dragomir et al. 1995). Let $f\in Q(I)$, $a,b\in I$, *with a <b* and $f\in L_1([a,b])$. Then

$$f\left(\frac{a+b}{2}\right)\le\frac{4}{b-a}\int_a^b f(x)\,dx. \tag{7}$$

Theorem 1.8 (Dragomir et al. 1995). Let $f\in P(I)$,$a,b\in I$, with $a<b$ and $f\in L_1([a,b])$. Then

$$f\left(\frac{a+b}{2}\right)\le\frac{2}{b-a}\int_a^b f(x)\,dx\le 2\,(f(a)+f(b)). \tag{8}$$

The following theorems were proved by Tunç (2010).

Theorem 1.9 (Tunc 2010). Let $a,b\in[0,\infty), a<b, I=[a,b]$ and *f, g*:$[a,b]\longrightarrow R$ be two functions and *f, g*$\in L_1[a,b]$. *If* $f\in Q(I)$ and $g\in P(I)$, then

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\le\frac{8(g(a)+g(b))}{b-a}\int_a^b f(x)\,dx\,. \tag{9}$$

Theorem 1.10 (Tunc 2010). Let $a,b\in[0,\infty), a<b, I=[a,b]$ with *f, g*:$[a,b]\longrightarrow R$ be two functions and *f, g*$\in L_1([a,b])$. *If f is convex and g belongs to the class P*(I) , then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx\le\frac{M(a,b)+N(a,b)}{2}\,, \tag{10}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$, $N(a,b)=f(a)g(b)+f(b)g(a)$.

Recently, Omotoyinbo and Mogbademu (2013), proved the following interesting results, independent on the recent work of Tunç et al. (2013).

Theorem 1.11 (Omotoyinbo and Mogbademu 2013). Let $a,b\in[0,\infty), a<b, I=[a,b]$ with *f,g*:$[a,b]\longrightarrow R$ be two functions and *f, g*$\in L_1([a,b])$. *If f*$\in Q(I)$ and $g\in MT$(I). Then the following inequality holds

$$\frac{1}{(b-a)}\int_a^b \mu(x)f(x)g(x)\,dx\le\left(\frac{3\pi}{128}\big(f(a)g(a)+f(b)g(b)\big)+\frac{5\pi}{128}\big(f(a)g(b)+f(b)g(a)\big)\right),$$

where $\mu(x)=\frac{(b-x)^2(x-a)^2}{(b-a)^4}$, $x\in[a,b]$. (11)

Theorem 1.12 (Omotoyinbo and Mogbademu 2013). Let *f, g*:$[a,b]\longrightarrow R$ be two functions with $a,b\in[0,\infty), a<b, I=[a,b]$ and *f, g*$\in L_1([a,b])$. If $f\in Q(I)$and $g\in MT(I)$. Then the following inequality holds

$$\frac{1}{2}\left(g(a)\frac{\sqrt{(b-x)^5(x-a)^3}}{(b-a)^5}\int_a^b f(x)dx+\ g(b)\frac{\sqrt{(b-x)^3(x-a)^5}}{(b-a)^5}\int_a^b f(x)dx\right)$$

$$+ f(a)\frac{(b-x)(x-a)^2}{(b-a)^4}\int_a^b g(x)dx + f(b)\frac{(b-x)^2(x-a)}{(b-a)^4}\int_a^b g(x)dx$$
$$\leq \frac{1}{2}\left(\frac{3\pi}{128}M(a,b) + \frac{5\pi}{128}N(a,b)\right) + \frac{(x-a)^2(b-x)^2}{(b-a)^5}\int_a^b f(x)g(x)dx, \quad (12)$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 1.13 (Omotoyinbo and Mogbademu 2013). Let *f, g*:$[a,b] \longrightarrow R$ be two functions with $a, b \in [0,\infty), a < b, I = [a,b]$ and *f, g*∈$L_1([a, b])$.*If f*∈ $Q(I)$ and *g*∈*MT*(*I*). Then, the following inequality holds;

$$\frac{32}{15\pi} fg\left(\frac{a+b}{2}\right) \leq M(a,b) + N(a,b), \quad (13)$$

where *M(a, b)* and *N*(*a,b*)are as given in Theorem 1.12 above.

Theorem 1.14 (Omotoyinbo and Mogbademu 2013). Let *f, g*:$[a,b] \longrightarrow R$ be two functions with $a, b \in [0,\infty), a < b, I = [a,b]$ and *f, g*∈$L_1([a, b])$.If $f \in Q(I)$ and *g*∈*MT(I).* Then

$$\frac{3\pi}{512} f\left(\frac{a+b}{2}\right)(g(a) + g(b)) + \frac{1}{12} g\left(\frac{a+b}{2}\right)(f(a) + f(b)) \leq \frac{1}{30} fg\left(\frac{a+b}{2}\right) + \frac{\pi}{64}(f(a)+f(b))(g(a) + g(b)) \quad (14)$$

Motivated by different works of these authors such as Dragomir et al. (1995), Tunç (2010), Omotoyinbo and Mogbademu (2013) and that of Tunç et al. (2013), we introduce and define some new concepts as follows:

Definition 1.15. A function *f:I* ⟶*R is m*-Godunova-Levin function or said to belong to the class *m-Q*(*I*) if *f* is nonnegative and for all $x, y \in I$, $m \in (0,1]$ and $t \in (0,1)$ satisfies the inequality;

$$f(tx + m(1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{m(1-t)}. \quad (15)$$

Definition 1.16. A function *f :I*⊆ *R*⟶ *R* is said to belong to the class *m-MT*(*I*) if it is nonnegative and $\forall x, y \in$ *I*, $m \in [0,1]$ and *t*∈ *(0,1)*, satisfies the inequality;

$$f(tx + m(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y) \quad . \quad (16)$$

Remark 1.17. Since $2\sqrt{t}\sqrt{1-t} \geq tm(1-t)$, *for* $t \in (0,1)$, $m \in (0,1]$, it is written

$$f(ta + m(1-t)b) \leq \frac{tf(a) + m(1-t)f(b)}{2\sqrt{t}\sqrt{1-t}} \leq \frac{tf(a) + m(1-t)f(b)}{tm(1-t)} .$$

As can be seen from this inequality, m-MT-convex functions class allows us to obtain a better upper bound than m-Godunova-Levin function. Obviously, m-$Q(I) \supset m$-$MT(I)$.

Remark 1.18

i) $f: [1,2] \longrightarrow \mathbb{R}$, $f(x) = x^p, p \in \left(0, \frac{1}{1000}\right)$.

For example, when we choose $I = [1,2],\ m = \left[\frac{1}{2},\ \frac{3}{2}\right],\ t = 0.5,\ x = 1,\ y = 2,$ then it is easy to see that inequalities (15) and (16) are true. Thus, f is *m*-Godunova-Levin convex and also *m-MT*-convex functions.

In this paper, using an analytical technique, we obtain some new integral inequalities of Hermite-Hadamard type involving two newly introduced convex functions; m-Godunova-Levin functions and m-MT-Convex functions.

MAIN RESULTS

Theorem 2.1.

Let $a, b \in [0,\infty), a < b, I = [a,b]$with *f, g*:$[a,b] \longrightarrow$ *R* be two functions and *f, g*∈L_1([*a, b*]). *If f*∈ m-$Q(I)$and *g*∈ m-$MT(I)$.*Then*

$$\frac{1}{(mb-a)}\int_a^{mb} \mu(x)f(x)g(x)\,dx \leq \frac{1}{2}\left(\frac{3m\pi}{128}\big(f(a)g(a)+f(b)g(b)\big)+\frac{5\pi}{128}\big(mf(a)g(b)+\ f(b)g(a)\big)\right),$$

where $\mu(x) = \frac{m(x-mb)^2(x-a)^2}{(mb-a)^4},\ x \in [a,b]$.

Proof: Since $f \in m$-$Q(I)$ and *g* is m-MT-convex, we have

$$f(ta+m(1-t)b) \leq \frac{1}{t}f(a)+\frac{1}{m(1-t)}f(b) \tag{17}$$

$$g(ta+m(1-t)b) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}g(a)+\frac{m\sqrt{1-t}}{2\sqrt{t}}g(b). \tag{18}$$

Multiplying (17) and (18), we get

$$f(ta+m(1-t)b)g(ta+m(1-t)b)$$
$$\leq \frac{1}{2}\left(\frac{\sqrt{t}\sqrt{1-t}}{t(1-t)}f(a)g(a)+\frac{m\sqrt{t}\sqrt{1-t}}{t^2}f(a)g(b)+\ \frac{\sqrt{t}\sqrt{1-t}}{m(1-t)^2}f(b)g(a)+\ \frac{\sqrt{t}\sqrt{1-t}}{t(1-t)}f(b)g(b)\right). \tag{19}$$

If both sides of inequality (19) are multiplied by $mt^2(1-t)^2$, we have

$$mt^2(1-t)^2\big(f(ta+m(1-t)b)g(ta+m(1-t)b)\big)$$
$$\leq \frac{1}{2}\big(f(a)g(a)mt(1-t)\sqrt{t}\sqrt{1-t}+\ f(a)g(b)m(1-t)^2\sqrt{t}\sqrt{1-t}$$
$$+f(b)g(a)t^2\sqrt{t}\sqrt{1-t}+f(b)g(b)mt(1-t)\sqrt{t}\sqrt{1-t}\big). \tag{20}$$

Integrating both sides of (20) with respect to$t$, we obtain

$$m\int_0^1 t^2(1-t)^2[f(ta+m(1-t)b)(g(ta+m(1-t)b)]dt$$
$$\leq\frac{1}{2}\Big(mf(a)g(a)\int_0^1 t(1-t)\sqrt{t}\sqrt{1-t}dt+\ mf(a)g(b)\int_0^1(1-t)^2\sqrt{t}\sqrt{1-t}dt$$
$$+f(b)g(a)\int_0^1 t^2\sqrt{t}\sqrt{1-t}dt+mf(b)g(b)\int_0^1 t(1-t)\sqrt{t}\sqrt{1-t}dt\Big). \quad (21)$$

Substituting $x=ta+(1-t)b$ and simplifying completely, equation (21) gives

$$\frac{m(mb-x)^2(x-a)^2}{(mb-a)^5}\int_a^b f(x)g(x)\,dx$$
$$\leq\frac{1}{2}\Big(\frac{3m\pi}{128}\big(f(a)g(a)+f(b)g(b)\big)+\frac{5\pi}{128}\big(mf(a)g(b)+f(b)g(a)\big)\Big).$$

The proof is completed.

Remark 2.2. If we set $m=1$ in Theorem 2.1, we obtain the following as corollary.

Corollary 2.3 (Omotoyinbo and Mogbademu 2013). Let $a,b\in[0,\infty),a<b,I=[a,b]$ with *f*, *g*:$[a,b]\longrightarrow$ *R* be two functions and *f*, *g*∈L_1 ([*a*, *b*]). If $f\in Q(I)$ and *g*∈*MT*(*I*). Then, the following inequality holds;

$$\frac{(b-x)^2(x-a)^2}{(b-a)^5}\int_a^b f(x)g(x)\,dx\leq\frac{1}{2}\Big(\frac{3\pi}{128}\big(f(a)g(a)+f(b)g(b)\big)+\frac{5\pi}{128}\big(f(a)g(b)+f(b)g(a)\big)\Big).$$

Theorem 2.4.

Let *f*, *g*:$[a,b]\longrightarrow$ *R* be two functions with $a,b\in[0,\infty),a<b,I=[a,b]$ *and f*, *g*∈L_1 ([*a*, *b*]).If *f*∈ m-$Q(I)$ and *g*∈ m-$MT(I)$. Then the following inequality holds

$$\frac{512}{15\pi}fg\left(\frac{a+b}{2}\right)\leq\frac{1}{m}[5m^2+6m+5]\big(M(a,b)+N(a,b)\big)+\ f(b)g(b)mt(1-t)\sqrt{t}\sqrt{1-t}\,,$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$,$N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof: Since *f*∈ m-$Q(I)$ and *g*∈ m-MT-convex, we can write

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{ta+m(1-t)b}{2}+\frac{m(1-t)a+tb}{2}\right)$$
$$\leq\frac{1}{2}(f(ta+m(1-t)b)+f(m(1-t)a+tb))$$
$$\leq\frac{1}{2}\left(\frac{1}{t}+\frac{1}{m(1-t)}\right)(f(a)+f(b)). \quad (22)$$

$$g\left(\frac{a+b}{2}\right)=g\left(\frac{ta+m(1-t)b}{2}+\frac{m(1-t)a+tb}{2}\right)$$
$$\leq\frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}}+\frac{m\sqrt{1-t}}{2\sqrt{t}}\right)(g(a)+g(b)). \quad (23)$$

By multiplying (22) and (23) together to have

$$fg\left(\frac{a+b}{2}\right) \leq \frac{1}{8}\left(\frac{1}{t}+\frac{1}{m(1-t)}\right)\left(\frac{\sqrt{t}\sqrt{1-t}}{(1-t)}+\frac{m\sqrt{t}\sqrt{1-t}}{t}\right)\times(f(a)+f(b))\,(g(a)+g(b)),$$
$$\leq \frac{1}{8}\left(\frac{m^2(1-t)^2\sqrt{t}\sqrt{1-t}+2mt(1-t)\sqrt{t}\sqrt{1-t}+t^2\sqrt{t}\sqrt{1-t}}{mt^2(1-t)^2}\right)+\ (f(a)f(b))(g(a)+g(b)). \quad (24)$$

If both sides of inequality (24) are multiplied by $mt^2(1-t)^2$, we obtain

$$mt^2(1-t)^2 fg\left(\frac{a+b}{2}\right) \leq \frac{1}{8}\big(f(a)+f(b)\big)\big(g(a)+g(b)\big)$$
$$\times\left(m^2(1-t)^2\sqrt{t}\sqrt{1-t}+\ 2mt(1-t)\sqrt{t}\sqrt{1-t}+t^2\sqrt{t}\sqrt{1-t}\right). \quad (25)$$

Integrating inequality (25) with respect to t over $[0,1]$, we get

$$mfg\left(\frac{a+b}{2}\right)\int_0^1 t^2\,(1-t)^2 dt \leq \frac{1}{8}(f(a)+f(b))(g(a)+g(b))$$
$$+\left(m^2\int_0^1 t^{\frac{1}{2}}\,(1-t)^{\frac{5}{2}}dt+2m\int_0^1 t^{\frac{3}{2}}(1-t)^{\frac{3}{2}}dt+\int_0^1 t^{\frac{5}{2}}(1-t)^{\frac{1}{2}}dt\right). \quad (26)$$

Further simplification of equation (26) gives

$$\frac{m}{30}fg\left(\frac{a+b}{2}\right) \leq \frac{\pi}{1024}[5m^2+6m+5]\big(f(a)+f(b)\big)\big(g(a)+g(b)\big).$$

Thus, the result is completed.

Remark 2.5. In Theorem 2.4, if *f* and *g* are similarly ordered, we obtain $\frac{256}{15\pi}fg\left(\frac{a+b}{2}\right) \leq \frac{1}{m}[5m^2+6m+5]M(a,b)$.

Theorem 2.6.

Let *f, g* :$[a,b] \longrightarrow$ *R* be two functions with $a,b \in [0,\infty), a < b, I[a,b]$ and *f, g*∈L_1 ([*a, b*]). If *f* ∈ m-$Q(I)$ and *g*∈ m-$MT(I)$.Then

$$\frac{3\pi}{1024}(m(m+1))\,f\left(\frac{a+b}{2}\right)(g(a)+g(b))+\frac{1}{24}(m+1)g\left(\frac{a+b}{2}\right)(f(a)+f(b))$$
$$\leq \frac{m}{30}fg\left(\frac{a+b}{2}\right)+\frac{\pi}{1024}[5m^2+6m+5]\,(f(a)+f(b))\,(g(a)+g(b)).$$

Proof: Since *f* ∈ m-$Q(I)$ and *g*∈ m-$MT(I)$, we can write

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left(\frac{1}{t}+\frac{1}{m(1-t)}\right)(f(a)+f(b)), g\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}}+\frac{m\sqrt{1-t}}{2\sqrt{t}}\right)(g(a)+g(b))$$

Recall that, for $p,q,r,s \in \mathbb{R}^+$, we have,$pq+rs \leq ps+qr$. Thus,

$$\frac{1}{2}f\left(\frac{a+b}{2}\right)\left(\frac{\sqrt{t}}{2\sqrt{1-t}}+\frac{m\sqrt{1-t}}{2\sqrt{t}}\right)(g(a)+g(b))+\frac{1}{\mathbf{2}}g\left(\frac{a+b}{2}\right)\left(\frac{1}{t}+\frac{1}{m(1-t)}\right)(f(a)+f(b))$$

$$\leq f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)+\left(\frac{1}{2}\left(\frac{1}{t}+\frac{1}{m(1-t)}\right)\left(f(a)+f(b)\right)\right)\left(\frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}}+\frac{m\sqrt{1-t}}{2\sqrt{t}}\right)\left(g(a)+g(b)\right)\right). \quad (27)$$

Simplifying (27) to get

$$\frac{1}{4}f\left(\frac{a+b}{2}\right)(g(a)+g(b))\left(\frac{m(1-t)\sqrt{t}\sqrt{1-t}+t\sqrt{t}\sqrt{1-t}}{t(1-t)}\right)+\frac{1}{2}g\left(\frac{a+b}{2}\right)(f(a)+f(b))\left(\frac{t+m(1-t)}{mt(1-t)}\right)$$

$$\leq fg\left(\frac{a+b}{2}\right)+\frac{1}{8}\left(\frac{m^2(1-t)^2\sqrt{t}\sqrt{1-t}+2mt(1-t)\sqrt{t}\sqrt{1-t}+t^2\sqrt{t}\sqrt{1-t}}{mt^2(1-t)^2}\right)(f(a)+f(b))(g(a)+g(b)). \quad (28)$$

Thus from equation (28), we have

$$\frac{1}{4}f\left(\frac{a+b}{2}\right)(g(a)+g(b))\left(m^2t(1-t)^2\sqrt{t}\sqrt{1-t}+mt^2(1-t)\sqrt{t}\sqrt{1-t}\right)$$

$$+\frac{1}{2}g\left(\frac{a+b}{2}\right)(f(a)+f(b))(mt(1-t)^2+t^2(1-t)),$$

$$\leq mt^2(1-t)^2fg\left(\frac{a+b}{2}\right)+\frac{1}{8}\left(m^2(1-t)^2\sqrt{t}\sqrt{1-t}+2mt(1-t)\sqrt{t}\sqrt{1-t}\right.$$

$$\left.+t^2\sqrt{t}\sqrt{1-t}\right)(f(a)+f(b))(g(a)+\left(g(b)\right) \quad (29)$$

Integrating equation (29) with respect to $t\ over\ [0.1]$,

$$\frac{1}{4}f\left(\frac{a+b}{2}\right)(g(a)+g(b))\int_0^1\left(m^2t(1-t)^2\sqrt{t}\sqrt{1-t}+mt^2(1-t)\sqrt{t}\sqrt{1-t}\right)dt$$

$$+\frac{1}{2}g\left(\frac{a+b}{2}\right)(f(a)+f(b))\int_0^1(mt(1-t)^2+t^2(1-t))dt$$

$$\leq mfg\left(\frac{a+b}{2}\right)\int_0^1 t^2(1-t)^2dt+\frac{1}{8}(f(a)+f(b))(g(a)+g(b))$$

$$\int_0^1\left(m^2(1-t)^2\sqrt{t}\sqrt{1-t}+2mt(1-t)\sqrt{t}\sqrt{1-t}+t^2\sqrt{t}\sqrt{1-t}\right)dt. \quad (30)$$

It is easy to see that, further simplification of equation (30) completely gives

$$\frac{3\pi}{1024}(m(m+1))f\left(\frac{a+b}{2}\right)(g(a)+g(b))+\frac{1}{24}(m+1)g\left(\frac{a+b}{2}\right)(f(a)+f(b))$$

$$\leq \frac{m}{30}fg\left(\frac{a+b}{2}\right)+\frac{\pi}{1024}[5m^2+6m+5]+\frac{\pi}{1024}[5m^2+6m+5](f(a)+f(b))(g(a)+g(b)).$$

Hence, the proof is completed.

Remark 2.7. Setting $m = 1$ in Theorem 2.6 above, we obtain the following corollary.
Corollary 2.8 (Omotoyinbo and Mogbademu 2013). Let $f, g:[a,b] \longrightarrow R$ be two functions with $a, b \in [0,\infty), a < b,\ I = [a,b]$ and $f, g \in L_1([a,b])$. If $f \in Q(I)$ and $g \in MT(I)$. Then

$$\frac{3\pi}{512} f\left(\frac{a+b}{2}\right)(g(a)+g(b)) + \frac{1}{12} g\left(\frac{a+b}{2}\right)(f(a)+f(b)) \leq \frac{1}{30} fg\left(\frac{a+b}{2}\right) + \frac{\pi}{64}(f(a)+f(b))(g(a)+g(b)).$$

APPLICATIONS TO MEANS

We now consider few applications of our results to the following special means of real numbers. Recall that, for any $a, b \in R$, Then

$$A = A(a,b) = \frac{a+b}{2}\ , a, b\ \geq 0,$$

$$G = G(a,b) = \sqrt{ab}\ ,\ \ a, b \geq 0\ ,$$

and

$$H = H(a,b) = \frac{2ab}{a+b}\ , \qquad a, b\ \geq 0,$$

are called the arithmetic mean, the geometric mean, and the harmonic mean respectively.
We get the following two propositions by applying our main results to these special means of real numbers.

Proposition 3.1

Let $f(x) = g(x) = x^p$, then

$$(A(a,b))^{2p} \leq \Upsilon G^2(a^p\ , b^p), \tag{31}$$

where $\gamma\ > 0,\ p\ \in \left(0,\ \frac{1}{1000}\right)$.

Proof: If we set $m = 1$in Theorem 2.1 and choose $x = \frac{a+b}{2}$, we obtain

$$\frac{(b-x)^2(x-a)^2}{(b-a)^4}\frac{1}{(b-a)}\int_a^b f(x)g(x)\,dx \leq \frac{1}{2}\left(\frac{3\pi}{128}\big(f(a)g(a)+f(b)g(b)\big) + \frac{5\pi}{128}\big(f(a)g(b)+\ f(b)g(a)\big)\right),$$

which gives

$$\frac{\left(\frac{b-a}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2}{(b-a)^4}\left(\frac{1}{b-a}\right)\left(\frac{a+b}{2}\right)^{2p}\int_a^b dx \leq \frac{1}{256}(3\pi(a^{2p}+b^{2p}) + 5\pi(2a^pb^p)). \tag{32}$$

Simplifying (32), we have

$$\frac{1}{16}\left(\frac{a+b}{2}\right)^{2p} \leq \frac{1}{256}(3\pi(a^{2p}+b^{2p}) + 5\pi(2a^pb^p))\ . \tag{33}$$

By recalling and substituting the following standard equalities in (33)

$$A(a,b)=\frac{a+b}{2}\Longrightarrow (A(a,b))^{2p}=\left(\frac{a+b}{2}\right)^{2p},$$

$A(a^{2p},b^{2p})=\frac{a^{2p}+b^{2p}}{2}\Longrightarrow 2A(a^{2p},b^{2p})=a^{2p}+b^{2p}$,

$G(a,b)=\sqrt{ab}\Longrightarrow G^2(a^p,b^p)=a^pb^p$,

we have

$$(A(a,b))^{2p}\le \frac{1}{16}\left(3\pi\cdot 2A(a^{2p},b^{2p})+5\pi\cdot 2G^2(a^p,b^p)\right) \quad (34)$$

$$\le \frac{1}{16}\left(3\pi\left(A(a,b)\right)^{2p}+10\pi G^2(a^p,b^p)\right).$$

Collecting like terms and simplifying completely gives

$$(A(a,b))^{2p}\left(1-\frac{3\pi}{16}\right)\le \frac{10\pi}{16}\left(G^2(a^p,b^p)\right),$$

that is,

$$(A(a,b))^{2p}\le \Upsilon G^2(a^p,b^p),$$

where $\Upsilon=\frac{10\pi}{16-3\pi}>0$.

Hence, the proof is completed.

Proposition 3.2

Let $f(x)=g(x)=\frac{1}{x^p}$, then

$(H(a,b))^{2p}\le \pi A(a^{2p},b^{2p})$,

where $p\in(0,\frac{1}{1000})$ and π a constant.

Proof: If we set $m=1$ in Theorem 2.1 and choose $x=\frac{a+b}{2}$, we obtain

$$\frac{(b-x)^2(x-a)^2}{(b-a)^4}\frac{1}{(b-a)}\int_a^b f(x)g(x)\,dx$$

$$\le \frac{1}{2}\left(\frac{3\pi}{128}\left(f(a)g(a)+f(b)g(b)\right)+\frac{5\pi}{128}\left(f(a)g(b)+f(b)g(a)\right)\right),$$

which gives

$$\frac{\left(\frac{b-a}{2}\right)^2+\left(\frac{b-a}{2}\right)^2}{(b-a)^4}\left(\frac{1}{b-a}\right)\frac{1}{\left(\frac{a+b}{2}\right)^{2p}}\int_a^b dx\le \frac{1}{256}\left(3\pi\left(\frac{1}{a^{2p}}+\frac{1}{b^{2p}}\right)+5\pi\left(\frac{1}{a^pb^p}+\frac{1}{b^pa^p}\right)\right). \quad (35)$$

Simplifying (35), we have

$$\frac{1}{\left(\frac{a+b}{2}\right)^{2p}} \leq \frac{1}{16}\left(\frac{3\pi(a^{2p}+b^{2p})+5\pi\cdot 2(ab)^{p}}{(ab)^{2p}}\right).$$

On further simplification, we get

$$\left(\frac{2ab}{a+b}\right)^{2p} \leq \frac{1}{16}(3\pi(a^{2p}+b^{2p})+5\pi\cdot 2(ab)^{p}). \quad (36)$$

Substituting into inequality (36), the relation

$0 \leq (a-b)^2 \Rightarrow ab \leq \frac{a^2+b^2}{2} \Rightarrow (ab)^p \leq \frac{a^{2p}+b^{2p}}{2}$,

We obtain

$\left(\frac{2ab}{a+b}\right)^{2p} \leq \pi\left(\frac{a^{2p}+b^{2p}}{2}\right)$,

Implying that

$(H(a,b))^{2p} \leq \pi A(a^{2p}, b^{2p})$.

Hence, the result is completed.

CONCLUSION

Several refinements, extensions and generalizations of the Hermite-Hadamard integral inequality for different classes of convex functions have been made in the recent years. This paper however borders on further extension and refinement of the double inequality for two newly introduced classes of convex functions; m-Godunova-Levin and m-MT-convex functions. A few number of results for special means of real numbers are also deduced for applications.

REFERENCES

Bakula MK, Pečarić J, Ribicic M. 2006. Companion inequalities to Jensen's inequality for *m*-convex and (□, *m*)-convex functions. J Inequal Pure Appl Math 7(5), Art. (194).

Dragomir SS, Pečarić J, Persson LE. 1995. Some inequalities of Hadamard type, Soochow J Math 21, 335-241.

Godunova EK, Levin VI. 1985. Neravenstva dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkcii. Vycislitel. Mat. i Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI, Moscow, pp. 138–142.

Hadamard J. 1893. Etude Sur les properties des fontionsentiereset en particular d'une function consideree par Riemann. J Math Pure Appl 58, 171-215.

Omotoyinbo O, Mogbademu A. 2013. On some Hadamard inequalities for Godunova-Levin and MT-Convex functions. J Nigerian Assoc. Math Phys 25(II):215-222.

Omotoyinbo O. 2014. Hermite-Hadamard Integral Inequalities for new kinds of convex functions. M. Sc. Thesis, University of Lagos, Lagos State, Nigeria.

Shi De-P, Bo-Yan X, Feng Q. 2014. Hermite-Hadamard type inequalities for (m, h_1, h_2) convex functions via Riemann-Liouville fractional Integrals. Turkish J Anal Numb Theory. 2(1): 23-28.

Toader GH. 1988. On a generalization of the convexity. Mathematica, 30(53):83-87.

Tunç M, Subas Y, Karabayir I. 2013. On some Hadamard type inequalities for MT-Convex functions. Int J Open Problems Comp Math 6(2):1998-6262.
Tunc M. 2010. On some Hadamard type inequalities for product of different kinds of convex functions, RGMIA, Res. Rep. Coll., Article 5, Available form: http//ajmaa.org,\RGMIA/ v13nl.php,13.
Tunç M.2012. On some new inequalities for convex functions. Turk J Math 36: 245-251.
Varošanec S. 2007. On h-convexity. J Math Anal Appl 326, 303-311.